Multiple Quantifiers

Lecture 11
Section 3.3

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Outline

Multiple Quantifiers

- Multiple universal statements
  - $\forall x \in S, \forall y \in T, P(x, y)$
  - $\forall y \in T, \forall x \in S, P(x, y)$
  - The order *does not* matter.
- Multiple existential statements
  - $\exists x \in S, \exists y \in T, P(x, y)$
  - $\exists y \in T, \exists x \in S, P(x, y)$
  - The order *does not* matter.

Multiple Universal Quantifiers

- Let $S = \{a, b, c\}$ and $T = \{r, s, t\}$.
- Then

$$\begin{aligned}
\forall x \in S, \forall y \in T, P(x, y) & \\
&\equiv \forall x \in S, [P(x, r) \wedge P(x, s) \wedge P(x, t)] \\
&\equiv [P(a, r) \wedge P(a, s) \wedge P(a, t)] \wedge [P(b, r) \wedge P(b, s) \wedge P(b, t)] \\
&\qquad \wedge [P(c, r) \wedge P(c, s) \wedge P(c, t)] \\
&\equiv [P(a, r) \wedge P(b, r) \wedge P(c, r)] \wedge [P(a, s) \wedge P(b, s) \wedge P(c, s)] \\
&\qquad \wedge [P(a, t) \wedge P(b, t) \wedge P(c, t)] \\
&\equiv \forall y \in T, [P(a, y) \wedge P(b, y) \wedge P(c, y)] \\
&\equiv \forall y \in T, \forall x \in S, P(x, y).
\end{aligned}$$

Mixed Quantifiers

- Mixed universal and existential statements
  - $\forall x \in S, \exists y \in T, P(x, y)$
  - $\exists y \in T, \forall x \in S, P(x, y)$
- The order *does* matter.
- What is the difference?

Mixed Quantifiers

- Let $S = \{a, b, c\}$ and $T = \{r, s, t\}$.
- Then

$$\begin{aligned} &\forall x \in S, \exists y \in T, P(x, y) \\ &\qquad \equiv \forall x \in S, [P(x, r) \vee P(x, s) \vee P(x, t)] \\ &\qquad \equiv [P(a, r) \vee P(a, s) \vee P(a, t)] \wedge [P(b, r) \vee P(b, s) \vee P(b, t)] \\ &\qquad\qquad \wedge [P(c, r) \vee P(c, s) \vee P(c, t)]. \end{aligned}$$

and

$$\begin{aligned} &\exists y \in T, \forall x \in S, P(x, y) \\ &\qquad \equiv \exists y \in T, [P(a, y) \wedge P(b, y) \wedge P(c, y)] \\ &\qquad \equiv [P(a, r) \wedge P(b, r) \wedge P(c, r)] \vee [P(a, s) \wedge P(b, s) \wedge P(c, s)] \\ &\qquad\qquad \vee [P(a, t) \wedge P(b, t) \wedge P(c, t)]. \end{aligned}$$

Mixed Quantifiers

- Let $S = \{a, b, c\}$ and $T = \{r, s, t\}$.
- Then

$$
\begin{aligned}
\forall x \in S, \exists y \in T, P(x, y) \\
&\equiv \forall x \in S, [P(x, r) \vee P(x, s) \vee P(x, t)] \\
&\equiv [P(a, r) \vee P(a, s) \vee P(a, t)] \wedge [P(b, r) \vee P(b, s) \vee P(b, t)] \\
&\qquad \wedge [P(c, r) \vee P(c, s) \vee P(c, t)].
\end{aligned}
$$

and

$$
\begin{aligned}
\exists y \in T, \forall x \in S, P(x, y) \\
&\equiv \exists y \in T, [P(a, y) \wedge P(b, y) \wedge P(c, y)] \\
&\equiv [P(a, r) \wedge P(b, r) \wedge P(c, r)] \vee [P(a, s) \wedge P(b, s) \wedge P(c, s)] \\
&\qquad \vee [P(a, t) \wedge P(b, t) \wedge P(c, t)].
\end{aligned}
$$

Mixed Quantifiers

- The order matters because, in the first case, the choice of y depends on the value of x.
- In the second case, the choice of y does not depend on the value of x.

Outline

Example

- What is the difference between the following two statements?

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0.$$

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y = 0.$$

- Are both statements true?

Example

- Consider the following two statements?

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 0.$$

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 0.$$

- Are both statements true?

Example

- Which of the following are true?

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, x(y+z) = 0.$$

$$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, x(y+z) = 0.$$

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall z \in \mathbb{R}, x(y+z) = 0.$$

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}, x(y+z) = 0.$$

- Write the negations of the ones that are false.

Example

- Consider the statements

$$\forall m \in \mathbb{R}, \forall b \in \mathbb{R}, \text{ the line } y = mx + b \text{ has an } x\text{-intercept.}$$

$$\forall m \in \mathbb{R}, \exists b \in \mathbb{R}, \text{ the line } y = mx + b \text{ has an } x\text{-intercept.}$$

$$\exists m \in \mathbb{R}, \forall b \in \mathbb{R}, \text{ the line } y = mx + b \text{ has an } x\text{-intercept.}$$

$$\exists m \in \mathbb{R}, \exists b \in \mathbb{R}, \text{ the line } y = mx + b \text{ has an } x\text{-intercept.}$$

- Which statements are true?
- How would you prove them?
- What are the negations of the false statements?

Example

- Suppose that a function $f(x)$ never gets but so large.
- Express that idea using quantifiers.
- Express its negation using quantifiers.

Example

- Suppose that we have two functions $f(x)$ and $g(x)$ such that no matter how large $f(x)$ gets, $g(x)$ at some point gets even larger.
- Express that idea using quantifiers.
- Express its negation using quantifiers.

Example

- Suppose that we have two functions $f(x)$ and $g(x)$ such that no matter how large $f(x)$ gets, $g(x)$ at some point gets even larger, *and vice versa*.
- Express that idea using quantifiers.
- Express its negation using quantifiers.

Outline

Assignment

Assignment

- Read Section 3.3, pages 117 - 128.
- Exercises 3, 9, 10, 12, 18, 19, 21, 29, 30, 55 - 58, page 129.